JOURNAL OF APPROXIMATION THEORY 7, 226–238 (1973)

Cesàro Summability of Laguerre Series

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Communicated by Yudell L. Luke

Received October 27, 1970

1. Let $f(x)$ be a function defined and Lebesgue measurable in the interval $[0, \infty]$ such that

$$\int_0^\infty e^{-u} u^\alpha f(u)\, du < \infty, \qquad \alpha > -1, \tag{1.1}$$

and

$$\int_0^\infty e^{-u} u^\alpha f(u)\, L_n^{(\alpha)}(u)\, du < \infty, \qquad \alpha > -1, \tag{1.2}$$

where $L_n^{(\alpha)}(x)$ denote the Laguerre functions of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^\infty L_n^{(\alpha)}(x)\, \omega^n = (1 - \omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right). \tag{1.3}$$

The Fourier–Laguerre expansion associated with the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^\infty a_n L_n^{(\alpha)}(x), \tag{1.4}$$

where

$$\Gamma(\alpha + 1) \binom{n+\alpha}{n} a_n = \int_0^\infty e^{-x} x^\alpha f(x)\, L_n^{(\alpha)}(x)\, dx. \tag{1.5}$$

In this paper we shall deal with the problem of Cesàro summability of the Fourier–Laguerre series (1.4) at the end point $x = 0$ of the linear interval $[0, \infty]$. Kogbetliantz [2, 3, 4] and Szegö [5, 6] have given a number of far-reaching results on the Cesàro-summability of series (1.4). A summability theorem for $x = 0$ given by Szegö reads as follows.

THEOREM A. *Let $f(x)$ be continuous at $x = 0$. If we assume the existence of the integral*

$$\int_1^\infty e^{-x/2} x^{\alpha-k-1/3} \, |f(x)| \, dx, \tag{1.6}$$

the Laguerre series (1.4) *is* (C, k)*-summable at* $x = 0$, *with the sum* $f(0)$, *provided* $k > \alpha + \frac{1}{2}$.

We shall prove certain generalizations of Theorem A. Our line of work corresponds to the classical work of Verblunsky [7] and Bosanquet [1] on Fourier trigonometric series.

The results on trigonometric series are so well known that I don't propose to restate them over here. However, the analogy will become clear as soon as we state our own results on Laguerre series.

We write

$$\phi(u) = [f(u) - f(0)] \frac{e^{-u} u^\alpha}{\Gamma(\alpha + 1)};$$

$$\Phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x - t)^{p-1} \phi(t) \, dt; \qquad p > 0; \tag{1.7}$$

$$\Phi_0(x) = \phi(x);$$

$$\phi_p(x) = \Gamma(p + 1) \, x^{-p} \Phi_p(x), \qquad p \geqslant 0;$$

and, clearly

$$\Phi_p(x) = \frac{d}{dx} \Phi_{p+1}(x), \qquad -1 < p < 0.$$

The following theorems will be proved.

THEOREM 1. *If*

$$F(t) \equiv \int_0^t | \phi(u)| \, du = o(t^{1+\alpha}), \tag{1.8}$$

and

$$\int_1^\infty e^{t/2} t^{-k-1/3} \, | \phi(t)| \, dt < \infty, \tag{1.9}$$

then the series (1.4) *is* (C, k)*-summable at the point* $x = 0$ *with the sum* $f(0)$, *provided that* $k > \alpha + \frac{1}{2}$ *and* $\alpha > -1$.

THEOREM 2. *If*

$$\int_0^t | \phi_p(u)| \, du = o(t^{1+\alpha}), \qquad p \geqslant 0, \tag{1.10}$$

and the condition (1.9) *holds true, then the series* (1.4) *is* (C, k)*-summable at the point* $x = 0$, *with the sum* $f(0)$, *provided that* $\alpha > -1$ *and* $k > \alpha + p + \frac{1}{2}$.

THEOREM 3. *If*

$$\int_0^t |\phi_p(u)|\, du = O(t^{1+\alpha}), \qquad p \geqslant 0, \tag{1.11}$$

$$\phi_{p+1}(t) = o(t^{\alpha}) \tag{1.12}$$

and condition (1.9) *holds true, then the series* (1.4) *is* (C, k)*-summable at the point* $x = 0$, *with the sum* $f(0)$, *provided that* $\alpha > -1$ *and* $k > \alpha + p + \frac{1}{2}$.

2. In order to prove the above theorems we shall make use of the following well known order estimates of Laguerre functions:

If α is arbitrary and real, C and ω are fixed positive constants, then as $n \to \infty$,

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4}), & \text{if} \quad c/n \leqslant x \leqslant \omega, \\ O(n^{\alpha}), & \text{if} \quad 0 \leqslant x \leqslant c/n. \end{cases} \tag{2.1}$$

For $\alpha \geqslant -\frac{1}{2}$, $0 < x \leqslant \omega$,

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4}), \\ O(n^{\alpha}), \end{cases} \tag{2.2}$$

both bounds being valid.

For $\alpha \leqslant -\frac{1}{2}$, $0 \leqslant x \leqslant \omega$,

$$L_n^{(\alpha)}(x) = O(n^{\alpha/2-1/4}). \tag{2.3}$$

If α and λ are arbitrary and real, $a > 0$, $0 < \eta < 4$, then for $n \to \infty$,

$$\max e^{-x/2}x^{\lambda} \mid L_n^{(\alpha)}(x)| \sim n^{Q}, \tag{2.4}$$

where

$$Q = \begin{cases} \max(\lambda - 1/2, \alpha/2 - 1/4), & a \leqslant x \leqslant (4 - \eta)\, n; \\ \max(\lambda - 1/3, \alpha/2 - 1/4), & x \geqslant a, \end{cases} \tag{2.5}$$

the maxima being taken in the intervals pointed out in the right hand members of (2.5).

A systematic derivation of all the above estimates has been made by Szegö [6, pp. 174–176].

3. *Cesàro means.* The n-th Cesàro sum of order k of the series

$$\sum_n L_n^{(\alpha)}(t), \tag{3.1}$$

is by definition (1.3), the coefficient of r^n in the expression

$$(1 - r)^{-k-1} \sum_{n=0}^{\infty} L_n^{(\alpha)}(t)\, r^n = (1 - r)^{-k-1} (1 - r)^{-\alpha-1} \exp\left(- \frac{x\omega}{1 - \omega}\right), \tag{3.2}$$

and is therefore equal to $L_n^{(\alpha+k+1)}(t)$.

The Laguerre series at $x = 0$ may be written as

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(0) = \{\Gamma(\alpha + 1)\}^{-1} \sum_{n=0}^{\infty} \int_0^{\infty} e^{-t} t^{\alpha} f(t)\, L_n^{(\alpha)}(t)\, dt. \tag{3.3}$$

On account of (3.2), the Cesàro means of order k are given by (see Szegö [6, p. 270]

$$\sigma_n^{(k)}(0) = \{A_n^{(k)} \Gamma(\alpha + 1)\}^{-1} \int_0^{\infty} e^{-t} t^{\alpha} f(t)\, L_n^{(\alpha+k+1)}(t)\, dt. \tag{3.4}$$

Writing $f(t) = f(0)$ in the n-th partial sum of the series (3.3) and employing the orthogonality property of Laguerre functions, we have

$$\begin{aligned} \sigma_n^{(k)}(0) - f(0) &= \{A_n^{(k)} \Gamma(\alpha + 1)\}^{-1} \int_0^{\infty} e^{-t} t^{\alpha} \{f(t) - f(0)\}\, L_n^{(\alpha+k+1)}(t)\, dt \\ &= \{A_n^{(k)}\}^{-1} \int_0^{\infty} \phi(t)\, L_n^{(\alpha+k+1)}(t)\, dt. \end{aligned} \tag{3.5}$$

4. *Proof of Theorem* 1. In order to prove the theorem we have to show that

$$\begin{aligned} I &\equiv \{A_n^{(k)}\}^{-1} \int_0^{\infty} \phi(t)\, L_n^{(\alpha+k+1)}(t)\, dt \\ &= o(1), \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{4.1}$$

We break the integral into three parts and write

$$I = \int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\omega} + \int_{\omega}^{\infty}, \tag{4.2}$$

(where ω is large but fixed, and η is suitably chosen small constant)

$$= I_1 + I_2 + I_3 + I_4\,, \text{ e.g.}$$

We first dispose of I_4 in the manner it is done by Szegö [6, p. 270]. In fact the integrability condition (1.9) has been assumed in such a way that I_4 may tend to 0 as $n \to \infty$. We apply the order estimate (2.4) with α replaced by

$\alpha + k + 1$, and $\lambda = k + \frac{1}{3}$, and we see that $\lambda - \frac{1}{3} = k > ((\alpha + k + 1)/2) - \frac{1}{4}$, whence

$$
\begin{aligned}
| I_4 | &= O(n^{-k}) \int_{\omega}^{\infty} | \phi(u)| \, e^{u/2} u^{-k-1/3} n^k \, du \\
&= O(1) \int_{\omega}^{\infty} | \phi(u)| \, e^{u/2} u^{-k-1/3} \, du \\
&= o(1),
\end{aligned}
\tag{4.3}
$$

since ω can be chosen as large as we please.

Coming to I_1, we have, using the order estimate (2.1)

$$
\begin{aligned}
| I_1 | &= O(n^{-k}) \int_0^{1/n} | \phi(u)| \, | L_n^{(\alpha+k+1)}(u)| \, du \\
&= O(n^{-k}) \int_0^{1/n} | \phi(u)| \, O(n^{\alpha+k+1}) \, du \\
&= O(n^{\alpha+1}) \int_0^{1/n} | \phi(u)| \, du \\
&= O(n^{\alpha+1}) \, o(1/n)^{1+\alpha}, \text{ using (1.8)}, \\
&= o(1), \qquad \text{as} \quad n \to \infty.
\end{aligned}
\tag{4.4}
$$

In I_2, we make use of the first order estimate in (2.1). We therefore have

$$
\begin{aligned}
| I_2 | &= O(n^{-k}) \left| \int_{1/n}^{\eta} \phi(t) \, L_n^{(\alpha+k+1)}(t) \, dt \right| \\
&= O(n^{-k}) \int_{1/n}^{\eta} | \phi(t)| \, t^{-(\alpha+k+1)/2-1/4} n^{(\alpha+k+1)/2-1/4} \, dt \\
&= O(n^{\alpha/2-k/2+1/4}) \int_{1/n}^{\eta} | \phi(t)| \, t^{-\alpha/2-k/2-3/4} \, dt \\
&= O(n^{\alpha/2-k/2+1/4}) \Big\{ [F(t) \, t^{-\alpha/2-k/2-3/4}]_{1/n}^{\eta} \\
&\quad + c \int_{1/n}^{\eta} F(t) \, t^{-\alpha/2-k/2-7/4} \, dt \Big\} \\
&= O(n^{\alpha/2-k/2+1/4}) \Big[F(\eta) \, \eta^{-\alpha/2-k/2-3/4} \\
&\quad + F\left(\frac{1}{n}\right) n^{\alpha/2+k/2+3/4} + \int_{1/n}^{\eta} o(t^{1+\alpha}) \, t^{-\alpha/2-k/2-7/4} \, dt \Big] \\
&= O(n^{\alpha/2-k/2+1/4}) + O(n^{\alpha/2+k/2+3/4}) \, o(1/n)^{1+\alpha} \, n^{\alpha/2-k/2+1/4} \\
&\quad + O(n^{\alpha/2-k/2+1/4}) \, o(n^{-\alpha/2+k/2-1/4}) \\
&= o(1) + o(1) + o(1), \text{ since } k > \alpha + \tfrac{1}{2}, \\
&= o(1).
\end{aligned}
\tag{4.5}
$$

Finally,

$$
\begin{aligned}
| I_3 | &= O(n^{-k}) \int_\eta^\omega | \phi(t)| \, | L_n^{(\alpha+k+1)}(t)| \, dt \\
&= O(n^{-k}) \int_\eta^\omega | \phi(t)| \, n^{(\alpha+k+1)/2-1/4} t^{-(\alpha+k+1)/2-1/4} \, dt \\
&= O(n^{-k/2+\alpha/2+1/4}) \int_\eta^\omega | \phi(t)| \, dt \\
&= o(1),
\end{aligned} \tag{4.6}
$$

since $\phi(t)$ is Lebesgue integrable and $k > \alpha + \frac{1}{2}$.

Combining (4.2),..., (4.6), we obtain the desired result.

5. *Proof of Theorem* 2. We have to show that

$$
I \equiv \frac{1}{A_n^{(k)}} \int_0^\infty \phi(u) \, L_n^{(\alpha+k+1)}(u) \, du = o(1). \tag{5.1}
$$

Write

$$
\begin{aligned}
I &= \int_0^\eta + \int_\eta^\omega + \int_\omega^\infty \\
&= I_1 + I_2 + I_3 \text{, e.g.,}
\end{aligned} \tag{5.2}
$$

where, as in Section 4, ω is taken sufficiently large and fixed and η is taken sufficiently small but fixed.

I_3 is disposed off exactly as in the proof of Theorem 1. We now take I_1. Integrating by parts m times, we obtain

$$
\begin{aligned}
I_1 &= (A_n^{(k)})^{-1} \left\{ \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(u) \left(\frac{d}{du}\right)^{\rho-1} L_n^{(\alpha+k+1)}(u) \right]_0^\eta \right. \\
&\qquad \left. + (-1)^m \int_0^\eta \Phi_m(u) \left(\frac{d}{du}\right)^m L_n^{(\alpha+k+1)}(u) \, du \right\} \\
&= A + (-1)^m B, \text{ e.g.}
\end{aligned} \tag{5.3}
$$

It is known that (see Szegö [6, p. 101])

$$
\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{5.4}
$$

Therefore,

$$A = (A_n^{(k)})^{-1} \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(u) L_{n-\rho+1}^{(\alpha+k+\rho)}(u) \right]_0^\eta$$
$$= (A_n^{(k)})^{-1} \sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(\eta) L_{n-\rho+1}^{(\alpha+k+\rho)}(\eta). \tag{5.5}$$

But

$$\begin{aligned}(A_n^{(k)})^{-1} L_{n-\rho+1}^{(\alpha+k+\rho)}(\eta) &= O(n^{-k}) O(n^{(\alpha+k+\rho)/2-1/4}), \quad \text{from (2.1) if} \quad \eta > 1/n,\\ &= O(n^{\alpha/2-k/2+\rho/2-1/4})\\ &= O(n^{\alpha/2-1/2(\alpha+p+1/2+\epsilon)+\rho/2-1/4}), \qquad \text{where} \quad \epsilon > 0,\\ &= O(n^{(-p+\rho-1-\epsilon)/2})\\ &= o(1), \qquad \text{if} \quad \rho < p+1. \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) it is seen that

$$A = o(1), \qquad \text{if} \quad p + 1 > m. \tag{5.7}$$

Let us denote $(d/dt)^m L_n^{(\alpha+k+1)}(t)$ by $S_n^{(m)}(t)$.
If p is not integral, let $p = m + \delta$, $0 < \delta < 1$. Consider the integral

$$\begin{aligned} v &= (A_n^{(k)})^{-1} \int_0^\eta \Phi_p(\gamma) S_n^{(p)}(\gamma)\, d\gamma\\ &= \frac{(A_n^{(k)})^{-1}}{\Gamma(1-\delta)} \int_0^\eta \Phi_p(\gamma)\, d\gamma \int_\gamma^\eta (t-\gamma)^{-\delta} S_n^{(m+1)}(t)\, dt\\ &= \frac{1}{\Gamma(1-\delta)} \int_0^\eta S_n^{(m+1)}(t)\, dt \int_0^t (t-\gamma)^{-\delta} \Phi_p(\gamma)\, d\gamma, \end{aligned}$$

but

$$\begin{aligned} \Phi_p(\gamma) &= \Phi_{m+\delta}(\gamma)\\ &= \frac{1}{\Gamma(\delta)} \int_0^\gamma (\gamma - t)^{\delta-1} \Phi_m(t)\, dt, \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^t (t-\gamma)^{-\delta} \Phi_p(\gamma)\, d\gamma &= \frac{1}{\Gamma(\delta)} \int_0^t (t-\gamma)^{-\delta}\, d\gamma \int_0^\gamma (\gamma-u)^{\delta-1} \Phi_m(u)\, du\\ &= \frac{1}{\Gamma(\delta)} \int_0^t \Phi_m(u)\, du \int_u^t (t-\gamma)^{-\delta} (\gamma-u)^{\delta-1}\, d\gamma. \end{aligned}$$

The integral on the right transforms by the substitution $\gamma = u + (t - u)^{\nu}$, into

$$\int_0^1 (1 - \nu)^{-\delta} \nu^{\delta-1} \, d\nu = \frac{\Gamma(1 - \delta) \, \Gamma(\delta)}{\Gamma(1)}.$$

Thus

$$\int_0^t (t - \gamma)^{-\delta} \, \Phi_p(\gamma) \, d\gamma = \Gamma(1 - \delta) \int_0^t \Phi_m(u) \, du$$
$$= \Gamma(1 - \delta) \, \Phi_{m+1}(t).$$

Consequently, from (5.8)

$$v = \int_0^{\eta} \Phi_{m+1}(u) \, S_n^{(m+1)}(u) \, du \, (A_n^{(k)})^{-1}$$
$$= \left[\Phi_{m+1}(\eta) \, S_n^{(m)}(\eta) - \int_0^{\eta} \Phi_p(u) \, S_n^{(p)}(u) \, du\right] (A_n^{(k)})^{-1},$$

and therefore we may write in (5.3)

$$B = \int_0^{\eta} \Phi_m(u) \, S_n^{(m)}(u) \, du$$
$$= (A_n^{(k)})^{-1} \left[\Phi_{m+1}(\eta) \, S_n^{(m)}(\eta) - \int_0^{\eta} \Phi_p(u) \, S_n^{(p)}(u) \, du\right] \tag{5.9}$$
$$= o(1) - (A_n^{(k)})^{-1} \int_0^{\eta} \Phi_p(u) \, S_n^{(p)}(u) \, du,$$

by the argument advanced in (5.6), since $m < p$.

From (5.3), (5.6), and (5.9) it is evident that it now remains to show that the integral

$$J \equiv (A_n^{(k)})^{-1} \int_0^{\eta} u^p \phi_p(u) \, S_n^{(p)}(u) \, du = o(1). \tag{5.10}$$

Write

$$J = \int_0^{c/n} + \int_{c/n}^{\eta} \tag{5.11}$$
$$= J_1 + J_2, \text{ e.g}$$

Now

$$
\begin{aligned}
| J_1 | &= O(n^{-k}) \int_0^{c/n} u^p \, | \phi_p(u)| \, | L_{n-p}^{(\alpha+k+1+p)}(u)| \, du \\
&= O(n^{-k}) \int_0^{c/n} u^p \, | \phi_p(u)| \, O(n^{\alpha+k+1+p}) \, du \\
&= O(n^{\alpha+1+p}) \frac{1}{n^p} \int_0^{c/n} | \phi_p(u)| \, du \\
&= O(n^{\alpha+1}) \, o(1/n)^{\alpha+1}, \qquad \text{from (1.10)}, \\
&= o(1).
\end{aligned} \tag{5.12}
$$

Again

$$
\begin{aligned}
| J_2 | &\leqslant (A_n^{(k)})^{-1} \int_{c/n}^{\eta} u^p \, | \phi_p(u)| \, | L_{n-p}^{(\alpha+k+1+p)}(u)| \, du \\
&= O(n^{-k}) \int_{c/n}^{\eta} u^p \, | \phi_p(u)| \, u^{-\alpha/2-k/2-1/2-p/2-1/4} \\
&\qquad \times \, n^{\alpha/2+k/2+1/2+p/2-1/4}, \qquad \text{from (2.1)}, \\
&= O(n^{(\alpha-k+p)/2+1/4}) \int_{c/n}^{\eta} | \phi_p(u)| \, u^{(p-\alpha-k)/2-3/4} \, du \\
&= O(n^{-\epsilon/2}) \int_{c/n}^{\eta} | \phi_p(u)| \, u^{-\alpha-\epsilon/2-1} \, du,
\end{aligned}
$$

having written $k = \alpha + p + \frac{1}{2} + \epsilon$.

Integrating by parts and writing

$$
\psi(t) = \int_0^t | \phi_p(u)| \, du,
$$

we obtain

$$
\begin{aligned}
| J_2 | &= O(n^{-\epsilon/2})[u^{-\alpha-\epsilon/2-1}\psi(u)]_{c/n}^{\eta} \\
&\qquad + O(n^{-\epsilon/2}) \int_{c/n}^{\eta} u^{-\alpha-\epsilon/2-2}\psi(u) \, du \\
&= O(n^{-\epsilon/2}) + O(n^{-\epsilon/2}) \, O(n^{\alpha+\epsilon/2+1}) \, o\left(\frac{1}{n^{1+\alpha}}\right) \\
&\qquad + O(n^{-\epsilon/2}) \int_{c/n}^{\eta} o(u^{-\alpha-\epsilon/2-2+1+\alpha}) \, du,
\end{aligned}
$$

from (1.10), and because η is chosen sufficiently small,

$$
\begin{aligned}
&= o(1) + o(1) + o(n^{-\epsilon/2}) \int_{c/n}^{\eta} u^{-1-\epsilon/2} \, du \\
&= o(1) + o(n^{-\epsilon/2})[u^{-\epsilon/2}]_{c/n}^{\eta} \\
&= o(1) + o(1) \\
&= o(1).
\end{aligned} \tag{5.13}
$$

Finally we come to the integral

$$I_2 = \frac{1}{(A_n^{(k)})} \int_\eta^\omega \phi(u)\, L_n^{(\alpha+k+1)}(u)\, du.$$

Using the estimate (2.1), we have

$$\begin{aligned} | I_2 | &= O(n^{-k}) \int_\eta^\omega | \phi(u)|\, | L_n^{(\alpha+k+1)}(u)|\, du \\ &= O(n^{-k}) \int_\eta^\omega | \phi(u)|\, n^{(\alpha+k+1)/2-1/4} u^{-(\alpha+k+1)/2-1/4}\, du \\ &= O(n^{\alpha/2-k/2+1/4}) \int_\eta^\omega | \phi(u)|\, du \\ &= O(n^{\alpha/2-k/2+1/4}), \text{ by the property of the Lebesgue integral,} \\ &= o(1), \qquad \text{since} \quad k > \alpha + p + \tfrac{1}{2}\,. \end{aligned} \tag{5.14}$$

Combining (5.2), (5.7), (5.9), (5.11), (5.12), (5.13) and (5.14), we have the desired result.

6. *Proof of Theorem* 3. The proof of Theorem 2 holds upto the stage of (5.9). Actually, what we have to demonstrate now is that under conditions (1.11) and (1.12),

$$J \equiv (A_n^{(k)})^{-1} \int_0^\eta u^p \phi_p(u)\, S_n^{(p)}(u)\, du = o(1). \tag{6.1}$$

Let m be a fixed number sufficiently large such that $m/n < \eta$. Now we write

$$\begin{aligned} J &= \int_0^{m/n} + \int_{m/n}^\eta \\ &= J_1 + J_2\,, \text{ e.g.} \end{aligned} \tag{6.2}$$

Consider first J_1.

$$\begin{aligned} J_1 &= \{A_n^{(k)}\}^{-1} \int_0^{m/n} \Phi_p(u)\, S_n^{(p)}(u)\, du \\ &= \{A_n^{(k)}\}^{-1} \left[\Phi_{p+1}(m/n)\, S_n^{(p)}(m/n) - \int_0^{m/n} t^{p+1} \phi_{p+1}(t)\, S_n^{(p+1)}(t)\, dt\right] \\ &= J_{1,1} + J_{1,2}\,, \text{ e.g.} \end{aligned} \tag{6.3}$$

Now

$$\begin{aligned} |J_{1,1}| &= O(n^{-k})[(m/n)^{p+1} \mid \phi_{p+1}(m/n)| \mid L_{n-p}^{(\alpha+k+1+p)}(m/n)|] \\ &= O(n^{-k})[O(n^{-p-1})\, o(m/n)^{\alpha}\, O(m/n)^{-(\alpha+k+1+p)/2-1/4} \\ &\quad \times n^{(\alpha+k+1+p)/2-1/4}], \qquad \text{from (2.1)}, \\ &= o(1). \end{aligned} \tag{6.4}$$

Coming to $J_{1,2}$, we have

$$\begin{aligned} |J_{1,2}| &= \left|\{A_n^{(k)}\}^{-1} \int_0^{m/n} t^{p+1}\phi_{p+1}(t)\, S_n^{(p+1)}(t)\, dt\right| \\ &= O(n^{-k}) \int_0^{m/n} t^{p+1} o(t^{\alpha}) \mid L_{n-p-1}^{(\alpha+k+p+2)}(t)|\, dt \\ &= O(n^{-k})\, O(n^{\alpha+k+p+2}) \int_0^{m/n} o(t^{p+\alpha+1})\, dt \\ &= O(n^{\alpha+p+2})\, o(m/n)^{\alpha+p+2} \\ &= o(1). \end{aligned} \tag{6.5}$$

Again,

$$\begin{aligned} |J_2| &= O(n^{-k}) \int_{m/n}^{\eta} u^p \mid \phi_p(u)\, S_n^{(p)}(u)|\, du \\ &= O(n^{-k}) \int_{m/n}^{\eta} u^p \mid \phi_p(u)| \mid L_{n-p}^{(\alpha+p+k+1)}(u)|\, du \\ &= O(n^{-k}) \int_{m/n}^{\eta} u^p \mid \phi_p(u)|\, O(n^{(\alpha+p+k+1)/2-1/4} u^{-(\alpha+p+k+1)/2-1/4}\, du \\ &= O(n^{(\alpha+p-k)/2+1/4}) \int_{m/n}^{\eta} u^{-(\alpha-p+k+1)/2-3/4} \mid \phi_p(u)|\, du \\ &= O(n^{-\epsilon/2}) \int_{m/n}^{\eta} u^{-\alpha-1-\epsilon/2} \mid \phi_p(u)|\, du, \end{aligned}$$

setting $k = \alpha + p + \frac{1}{2} + \epsilon$.

Integrating by parts and writing

$$\psi(t) = \int_0^t | \phi_p(u)| \, du,$$

$$\begin{aligned} | J_2 | &= O(n^{-\epsilon/2})[u^{-\alpha-1-\epsilon/2}\psi(u)]_{m/n}^{\eta} \\ &\quad + O(n^{-\epsilon/2}) \int_{m/n}^{\eta} u^{-\alpha-\epsilon/2-2}\psi(u) \, du \\ &= J_{2,1} + J_{2,2}\text{, e.g.} \end{aligned} \tag{6.6}$$

$$\begin{aligned} |J_{2,1} | &= O(n^{-\epsilon/2}) + O(n^{-\epsilon/2})(m/n)^{-\alpha-\epsilon/2-1} \, O(m/n)^{\alpha+1} \\ &= o(1) + O(m^{-\epsilon/2}) \qquad (6.7) \\ &= o(1), \end{aligned}$$

if m is chosen sufficiently large.

Also

$$\begin{aligned} | J_{2,2} | &= O(n^{-\epsilon/2}) \int_{m/n}^{\eta} O(t^{1+\alpha}) \frac{dt}{t^{\alpha+\epsilon/2+2}} \\ &= O(n^{-\epsilon/2}) \int_{m/n}^{\eta} t^{-\epsilon/2-1} \, dt \\ &= O(n^{-\epsilon/2})[t^{-\epsilon/2}]_{m/n}^{\eta} \\ &= o(1) + O(m^{-\epsilon/2}) \\ &= o(1), \end{aligned} \tag{6.8}$$

as before by choosing n sufficiently large. Combining (6.1)–(6.8), we have the final result.

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